

CAT(0) EXTENSIONS OF RIGHT-ANGLED COXETER GROUPS

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ABSTRACT. We show that any split extension of a right-angled Coxeter group W_Γ by a generating automorphism of finite order acts faithfully and geometrically on a CAT(0) metric space.

1. INTRODUCTION

An isometric group action is *faithful* if its kernel is trivial, and it is *geometric* if it is cocompact and properly discontinuous. A finitely generated group G is a *CAT(0) group* if there exists a CAT(0) metric space X equipped with a faithful geometric G -action. The CAT(0) property is not an invariant of the quasi-isometry class of a group (see, for example, [1, 6] and [3, p. 258]). Whether or not it is an invariant of the abstract commensurability class of a group is as yet unknown. Attention was brought to this matter in [8]. In this article we illustrate that answering this question for any family of CAT(0) groups may require a variety of techniques.

It is well-known that an arbitrary right-angled Coxeter group W is a CAT(0) group because it acts faithfully and geometrically on a CAT(0) cube complex X . It is also well-known that the automorphism group $\text{Aut}(W)$ is generated by three types of finite-order automorphisms. As a natural source of examples we consider split extensions of right-angled Coxeter groups by finite cyclic groups, where in each case the cyclic group acts on W as the group generated by one of these various generating automorphisms. Our theorem is the following:

Theorem 1.1. *Suppose W is a right-angled Coxeter group and $\phi \in \text{Aut}(W)$ is either an automorphism induced by a graph automorphism,*

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a partial conjugation, or a transvection. Let m denote the order of ϕ . Then the group $G = W \rtimes_{\phi} \mathbb{Z}/m\mathbb{Z}$ is a CAT(0) group.

What is most interesting is that G is a CAT(0) group for different reasons in each of the three cases. When ϕ is an automorphism induced by a graph automorphism, the left-multiplication action $W \curvearrowright X$ extends to an action $G \curvearrowright X$; when ϕ is a partial conjugation, G is itself a right-angled Coxeter group; when ϕ is a transvection, G is not a right-angled Coxeter group and the action $W \curvearrowright X$ cannot extend to all of G , but we can explicitly construct a new CAT(0) space Y and describe a faithful geometric action $G \curvearrowright Y$.

After necessary background material is described in Section 2, the three cases of the theorem are treated, in turn, in Sections 3, 4 and 5.

We also note that, in each case of the theorem, we take an extension $W_{\Gamma} \rtimes H$ where $H \leq \text{Aut}(W_{\Gamma})$ is finite. In [4], we give an example in which H is infinite and $W_{\Gamma} \rtimes H$ is not a right-angled Coxeter group. We currently do not know whether such extensions with infinite H are CAT(0) or not. Since this question does not address the abstract commensurability of the CAT(0) property, we will not address it further in this paper.

2. RIGHT-ANGLED COXETER GROUPS AND THEIR AUTOMORPHISMS

In this section we briefly recall a very small part of the rich combinatorial and geometric theory of right-angled Coxeter groups. The interested reader may consult [5] for a thorough account of the more general subject of Coxeter groups from the geometric group theory point of view.

Fix an arbitrary finite simple graph Γ with vertex set S and edge set E . The *right-angled Coxeter group defined by Γ* is the group $W = W_{\Gamma}$ generated by S , with relations declaring that the generators all have order 2, and adjacent vertices commute with each other. The pair (W, S) is called a right-angled Coxeter system. As described in [5, Proposition 7.3.4, p. 130], we construct a cube complex $X = X(W, S)$ inductively as follows:

- The set of vertices is indexed by W , say $X^0 = \{v_w \mid w \in W\}$.
- To complete the construction of the one-skeleton X^1 we add edges of unit length so that vertices v_u, v_w are adjacent if and only if $u^{-1}w \in S$.
- For each $k \geq 2$, we construct the k -skeleton by gluing in Euclidean unit cubes of dimension k whenever X^{k-1} contains the $(k-1)$ -skeleton of such a cube.

Remark 2.1. We note the following about this construction:

- The dimension of X equals the number of vertices in the largest clique in Γ .

- The barycentric subdivision of X is the well-known Davis complex $\Sigma = \Sigma(W, S)$. By a result of Gromov, Σ , and hence also X , is a CAT(0) metric space (see [5, Theorem 12.3.3, p. 235] for a generalization due to Moussong).

By construction, the geometry of X is determined entirely by its 1-skeleton X^1 . It follows that a permutation σ of the vertex set X^0 determines an isometry of X if it respects the adjacency relation. In particular, for all $w \in W$ the map $v_u \mapsto v_{wu}$ extends to an isometry $\Phi_w \in \text{Isom}(X)$. The map $w \mapsto \Phi_w$ is a faithful geometric action $W \curvearrowright X$ known as the *left-multiplication action*.

From the graph Γ we may infer the existence of certain finite-order automorphisms of W . For each vertex $a \in S$, we write $\text{Lk}(a)$ for the set of vertices adjacent to a , and $\text{St}(a)$ for $\text{Lk}(a) \cup \{a\}$.

- Each graph automorphism $f \in \text{Aut}(\Gamma)$ restricts to a permutation of S which determines an automorphism $\phi_f \in \text{Aut}(W)$.
- For each union of non-empty connected components D of $\Gamma \setminus \text{St}(a)$, the map

$$s \mapsto \begin{cases} asa & s \in D, \\ s & s \in S \setminus D, \end{cases}$$

determines an automorphism of W called the *partial conjugation with acting letter a and domain D* .

- If $a, d \in S$ are such that $\text{St}(d) \subseteq \text{St}(a)$, then the rule

$$s \mapsto s \text{ for all } s \in S \setminus \{d\}, \quad d \mapsto da,$$

determines an automorphism of W called the *transvection with acting letter a and domain d* .

Together, the automorphisms induced by graph automorphisms, the partial conjugations and the transvections comprise a generating set for $\text{Aut}(W)$ [7]. We note that partial conjugations and transvections are involutions, and graph automorphisms have finite order.

In what follows, $\phi \in \text{Aut}(W)$ shall always denote a non-trivial automorphism of finite order m , and G shall denote the semi-direct product $G = W \rtimes_{\phi} \mathbb{Z}/m\mathbb{Z}$. So G is presented by:

$$P_1 = \langle S \cup \{z\} \mid s^2 = 1 \text{ for all } s \in S, [s, t] = 1 \text{ for all } \{s, t\} \in E, \\ z^m = 1, zsz^{-1} = \phi(s) \text{ for all } s \in S \rangle.$$

3. WHEN ϕ IS INDUCED BY A GRAPH AUTOMORPHISM

Suppose ϕ is induced by a graph automorphism $f \in \text{Aut}(\Gamma)$. Then the map $v_w \mapsto v_{\phi(w)}$ preserves the adjacency relation in X^1 , and hence determines an isometry $\Phi \in \text{Isom}(X)$. By simple computation the reader

may confirm that the relations in the presentation P_1 are satisfied when each $s \in S$ is replaced by Φ_s , and z is replaced by Φ . Hence the rule

$$s \mapsto \Phi_s \text{ for all } s \in S, z \mapsto \Phi,$$

determines an action $G \curvearrowright X$. We leave the reader to confirm that the action is faithful and geometric, and hence Theorem 1.1 holds in the first of the three cases.

In fact, a stronger result holds for similar reasons.

Lemma 3.1. *If $\mathcal{H} \leq \text{Aut}(\Gamma)$ is the group of graph automorphisms and H is the corresponding subgroup of $\text{Aut}(W)$, then the natural action $W \curvearrowright X$ extends to a faithful geometric action $W \rtimes H \curvearrowright X$.*

4. WHEN ϕ IS A PARTIAL CONJUGATION

Now suppose that ϕ is the partial conjugation with acting letter a and domain D . Recall that v_w denotes the vertex of X indexed by the group element $w \in W$. For any $d \in D$, v_1 and v_d are adjacent in X^1 , but $v_{\phi(1)}$ and $v_{\phi(d)}$ are not. Since the map $v_w \mapsto v_{\phi(w)}$ does not respect adjacency in X^1 , the left-multiplication action $W \curvearrowright X$ does not naturally extend to an action $G \curvearrowright X$. However, G is itself a right-angled Coxeter group, and hence also a CAT(0) group.

Lemma 4.1. *If ϕ is a partial conjugation with acting letter a and domain D , then G is itself a right-angled Coxeter group.*

We will omit the details of the proof, which may be found in [4]. In that paper, we engage more broadly with the problem of identifying a right-angled Coxeter presentation in a given group (or proving that no such presentation exists). We find various families of extensions of right-angled Coxeter groups which are again right-angled Coxeter, and these include Lemma 4.1 as a special case.

Here we will give a description of how to construct the defining graph Λ for G based on the original graph Γ . The procedure is as follows:

- (1) Add a new vertex labeled x , which we connect to everything in $\Gamma \setminus D$.
- (2) Replace the label of vertex a with the label ax , and add edges connecting ax to each vertex in D .

An example is shown in Figure 1.

5. WHEN ϕ IS A TRANSVECTION

Finally, we suppose that ϕ is the transvection with acting letter a and domain d . Recall that this means that $\text{St}(d) \subseteq \text{St}(a)$, and ϕ is determined

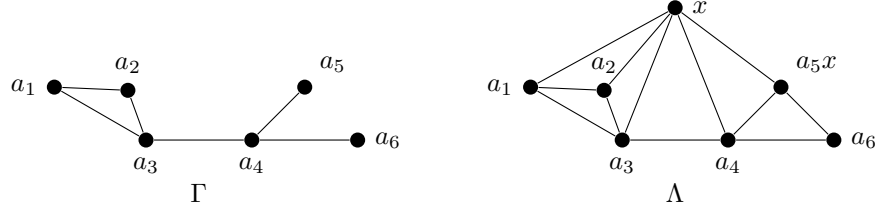


FIGURE 1. Λ is the defining graph of $W_\Gamma \rtimes \langle x \rangle$, where x has acting letter a_5 and domain $\{a_6\}$.

by the rule:

$$d \mapsto da, \text{ and } s \mapsto s \text{ for all } s \in S \setminus \{d\}.$$

We note that v_1 and v_d are adjacent in X^1 , but $v_{\phi(1)}$ and $v_{\phi(d)}$ are not. Since the map $v_w \mapsto v_{\phi(w)}$ does not respect adjacency in X^1 , the left-multiplication action $W \curvearrowright X$ does not naturally extend to an action $G \curvearrowright X$. In fact, a stronger statement is true. It follows from [5, Section 13.2] that $\text{Fix}(d)$ is a codimension 1 subspace of Σ , and $\text{Fix}(da)$ is codimension 2. Hence there is no isometry of X which can conjugate the isometry representing d to give the isometry representing da , so the left-multiplication action $W \curvearrowright X$ cannot be extended in any way to an action $G \curvearrowright X$.

We also note that G does not embed in a right-angled Coxeter group since G contains an element of order 4. Since $xdx = ad$, we have that $(xd)^2 = a$ and xd has order 4. In a right-angled Coxeter group, any non-trivial element of finite order is an involution.

It seems that to show that G is a CAT(0) group, we must identify a new CAT(0) space Y , and describe a faithful geometric action $G \curvearrowright Y$. The key to our success in doing exactly this is the existence of a certain finite-index subgroup of W which is itself a right-angled Coxeter group. Although the existence of such a subgroup is well-known (see [2, Example 1.4], for example, where the analogous subgroup is used in the context of right-angled Artin groups), we provide the details here for completeness.

Let $h_a : W \rightarrow \mathbb{Z}/2\mathbb{Z}$ denote the homomorphism determined by the rule: $a \mapsto 1$, and $s \mapsto 0$ for all $s \in S \setminus \{a\}$. Let U denote the kernel of h_a , and let

$$S' = (S \setminus \{a\}) \cup \{asa \mid s \in S \setminus \text{St}(a)\}.$$

Lemma 5.1. *The pair (U, S') is a right-angled Coxeter system, and hence U is a right-angled Coxeter group. Further, conjugation by a in W restricts to an automorphism $\theta \in \text{Aut}(U)$ induced by a permutation of S' ; this automorphism is trivial if and only if a is central in W .*

Proof. If a is central in W , then $S' = S \setminus \{a\}$, and the result is evident. In this case conjugation by a restricts to the trivial automorphism of U , and hence is the automorphism of U induced by the trivial permutation of S' .

Suppose a is not central in W . An alternative presentation for W may be constructed from the standard Coxeter presentation for W by the following Tietze transformations:

- For each vertex $s \in S \setminus \text{St}(a)$, introduce a new generator \hat{s} , the defining relation $asa = \hat{s}$, and redundant relations $a\hat{s}a = s$ and $\hat{s}^2 = 1$.
- For each pair of adjacent vertices $s, t \in S \setminus \text{St}(a)$, introduce the redundant relation $\hat{s}\hat{t} = \hat{t}\hat{s}$.
- For each pair of adjacent vertices $s \in S \setminus \text{St}(a)$ and $t \in \text{Lk}(a)$, introduce the redundant relation $\hat{s}t = t\hat{s}$.
- For each vertex $x \in \text{Lk}(a)$, we rewrite the relation $xa = ax$ as $axa = x$.

The resulting presentation of W is:

$$\begin{aligned} P_2 = \langle S' \cup \{a\} \mid & x^2 = 1 \text{ for all } x \in S', \\ & [s, t] = 1 \text{ for all } \{s, t\} \in E \text{ such that } s, t \neq a, \\ & [\hat{s}, \hat{t}] = 1 \text{ for all } \{s, t\} \in E \text{ such that } s, t \in S \setminus \text{St}(a), \\ & [\hat{s}, t] = 1 \text{ for all } \{s, t\} \in E \text{ such that } s \in S \setminus \text{St}(a) \text{ and } t \in \text{Lk}(a), \\ & a^2 = 1, asa = s \text{ for all } s \in \text{Lk}(a), \\ & asa = \hat{s} \text{ and } a\hat{s}a = s \text{ for all } s \in S \setminus \text{St}(a) \rangle. \end{aligned}$$

Evidently, this is the presentation of a semi-direct product in which the non-normal factor is $\langle a \rangle$, the normal factor is a right-angled Coxeter group with generating set

$$S' = (S \setminus \{a\}) \cup \{\hat{x} \mid x \in S \setminus \text{St}(a)\},$$

and a acts on the normal factor as the automorphism θ induced by permuting the generators according to the rule

$$x \mapsto \hat{x} \text{ and } \hat{x} \mapsto x \text{ for all } x \in S \setminus \text{St}(a), y \mapsto y \text{ for all } y \in \text{Lk}(a).$$

The action of a on U is non-trivial because $S \neq \text{St}(a)$. □

We now have the following refined decomposition of G :

$$G = (U \rtimes_{\theta} \langle a \rangle) \rtimes_{\phi} \langle z \rangle.$$

A presentation P_3 for G is obtained from the presentation P_2 for W by appending the generator z and relations

$$z^2 = 1, zsz = s \text{ for all } s \in S' \setminus \{d\}, zdz = da, zaz = a.$$

It follows that for each $g \in G$, there exist unique choices $u_g \in U$, and $\epsilon_g, \delta_g \in \{0, 1\}$, such that $g = u_g a^{\epsilon_g} z^{\delta_g}$. We shall write Y for the CAT(0) cube complex on which U acts geometrically and faithfully as defined in Section 2, and we write $p : G \rightarrow U$ for the projection map $g \mapsto u_g$. The projection map is not a homomorphism because for $s \in S \setminus \text{St}(a)$ we have $p(a)p(s)p(a) = s \neq s' = p(s')$. Even so, it allows us to parlay the left-multiplication action of G on itself into an action of $G \cup Y$.

Lemma 5.2. *For all $g \in S' \cup \{a, z\}$, the rule*

$$v_u \mapsto v_{p(gu)} \text{ for all } u \in U,$$

respects adjacency in Y^1 , and hence determines an isometry $\Phi_g \in \text{Isom}(Y)$.

Proof. Let $u \in U$, $s \in S'$ and $g \in S' \cup \{a, z\}$. To prove the result we must establish that $v_{p(gu)}$ and $v_{p(gus)}$ are adjacent. For this it suffices to show that $(p(gu))^{-1}p(gus) \in S'$.

If $g \in S'$, then

$$(p(gu))^{-1}p(gus) = (gu)^{-1}gus = s \in S'.$$

If $g = a$, then

$$\begin{aligned} (p(au))^{-1}p(aus) &= (p(\theta(u)a))^{-1}p(\theta(us)a) \\ &= (\theta(u))^{-1}\theta(us) \\ &= \theta(s) \in S'. \end{aligned}$$

Finally, we consider the case $g = z$. We note that if d occurs an even number of times in any word for u , then a occurs an even number of times in any word for $\phi(u)$, and $p(zu) = \phi(u)$. If, on the other hand, d occurs an odd number of times in any word for u , then a occurs an odd number of times in any word for $\phi(u)$, and $p(zu) = \phi(u)a$. The parity of d in a group element $u \in U$ is identified by the homomorphism $h_d : U \rightarrow \mathbb{Z}/2\mathbb{Z}$ determined by the rule $d \mapsto 1$, and $s \mapsto 0$ for all $s \in S' \setminus \{d\}$. Therefore, we consider cases based on the value of $h_d(u)$, and whether or not $s = d$.

If $h_d(u) = 0$ and $s \neq d$, then

$$(p(zu))^{-1}p(zus) = (\phi(u))^{-1}\phi(us) = s \in S'.$$

If $h_d(u) = 0$ and $s = d$, then

$$(p(zu))^{-1}p(zud) = (\phi(u))^{-1}\phi(ud)a = \phi(d)a = d \in S'.$$

If $h_d(u) = 1$ and $s \neq d$, then

$$(p(zu))^{-1}p(zus) = (\phi(u)a)^{-1}\phi(us)a = a\phi(u)^{-1}\phi(us)a = asa = \theta(s) \in S'.$$

If $h_d(u) = 1$ and $s = d$, then

$$(p(zu))^{-1}p(zud) = (\phi(u)a)^{-1}\phi(ud) = a\phi(u)^{-1}\phi(u)da = ada = d \in S'.$$

Adjacency is respected in all cases, so the result holds in the case that $g = z$, and thus Φ_g is an isometry of Y as required. \square

In summary, we have that G is presented by:

$$\begin{aligned}
P_3 = \langle S' \cup \{a, z\} \mid & x^2 = 1 \text{ for all } x \in S', \\
& [s, t] = 1 \text{ for all } \{s, t\} \in E \text{ such that } s, t \neq a, \\
& [\hat{s}, \hat{t}] = 1 \text{ for all } \{s, t\} \in E \text{ such that } s, t \in S \setminus \text{St}(a), \\
& [\hat{s}, t] = 1 \text{ for all } \{s, t\} \in E \text{ such that } s \in S \setminus \text{St}(a) \text{ and } t \in \text{Lk}(a), \\
& a^2 = 1, asa = s \text{ for all } s \in \text{Lk}(a), \\
& asa = \hat{s} \text{ and } a\hat{s}a = s \text{ for all } s \in S \setminus \text{St}(a), \\
& z^2 = 1, zsz = s \text{ for all } s \in S' \setminus \{d\}, zdz = da, zaz = a \rangle;
\end{aligned}$$

and

$$\begin{aligned}
\Phi_s(v_u) &= v_{su} \text{ for all } s \in S', \\
\Phi_a(v_u) &= v_{\theta(u)}, \\
\Phi_z(v_u) &= v_{\phi(u)} \text{ if } h_d(u) = 0, \\
\Phi_z(v_u) &= v_{\phi(u)a} \text{ if } h_d(u) = 1.
\end{aligned}$$

Lemma 5.3. *The map*

$$g \mapsto \Phi_g \text{ for all } g \in S' \cup \{a, z\},$$

determines a geometric action $G \curvearrowright Y$ which extends the left-multiplication action $U \curvearrowright Y$. If a is not central in W , the action is faithful. If a is central in W , the kernel is the subgroup generated by $\{a, z\}$.

Proof. To prove that the map determines an isometric group action, we must prove that the relations in the presentation P_3 for G hold when each $g \in S' \cup \{a, z\}$ is replaced by Φ_g . It is clear that those relations not involving either a or z remain true when each $g \in S'$ is replaced by Φ_g . We leave the reader to verify that the following relations hold (using the

rules listed immediately before the statement of the lemma):

$$\begin{aligned}
\Phi_a^2 &= 1, \\
\Phi_a \Phi_s \Phi_a &= \Phi_s \text{ for all } s \in \text{Lk}(a), \\
\Phi_a \Phi_s \Phi_a &= \Phi_{\hat{s}} \text{ for all } s \in S \setminus \text{St}(a), \\
\Phi_a \Phi_{\hat{s}} \Phi_a &= \Phi_s \text{ for all } s \in S \setminus \text{St}(a), \\
\Phi_z^2 &= 1, \\
\Phi_z \Phi_s \Phi_z &= \Phi_s \text{ for all } s \in S' \setminus \{d\}, \\
\Phi_z \Phi_d \Phi_z &= \Phi_d \Phi_a, \\
\Phi_z \Phi_a \Phi_z &= \Phi_a.
\end{aligned}$$

We note that, because $v_1 \mapsto v_{p(g)}$, the stabilizer of v_1 is a subgroup of the finite abelian group $\langle a, z \rangle$. If a is not central in W , there exists $s \in S \setminus \text{St}(a)$. Computation shows that Φ_a, Φ_{az} do not fix v_s , and Φ_z does not fix v_{ds} . Our claims about the kernel of the action follow immediately. \square

If a is central in W , then there is no obvious way in which a should act non-trivially on Y . We can, however, extend Y to a new space Y^+ by appending two unit length edges in a “v” shape at each vertex, thereby providing pieces on which a and ϕ can act non-trivially. More formally, to construct Y^+ from Y we write v_u^0 for v_u , and we append new vertices

$$\{v_u^i \mid \text{for all } u \in U \text{ and } i \in \{-1, 1\}\},$$

and new unit length edges

$$\{\{v_u^0, v_u^{-1}\}, \{v_u^0, v_u^1\} \mid \text{for all } u \in U\}.$$

It is evident that appending such “v” shapes at each vertex does not cause the CAT(0) property to fail, hence Y^+ is a CAT(0) cube complex.

Proposition 5.4. *If a is central in W , then G acts faithfully and geometrically on Y^+ .*

Proof. Suppose that a is central in W , i.e., that $\text{St}(a) = \Gamma$. Then (U, S') is a right-angled Coxeter system, and $W = U \times \langle a \rangle$.

We now define a homomorphism $\Phi: G \rightarrow \text{Isom}(Y^+)$. For each $s \in S'$, we declare $\Phi(s)$ to be the isometry determined by the rule:

$$v_u^i \mapsto v_{su}^i \text{ for all } u \in U \text{ and } i \in \{-1, 0, 1\}.$$

We declare $\Phi(a)$ to be the isometry determined by the rule:

$$v_u^i \mapsto v_u^{-i} \text{ for all } u \in U \text{ and } i \in \{-1, 0, 1\}.$$

We declare $\Phi(z)$ to be the isometry determined by the rule:

$$v_u^i \mapsto \begin{cases} v_u^i & \text{if } h_d(u) = 0, \\ v_u^{-i} & \text{if } h_d(u) = 1, \end{cases}$$

for all $u \in U$ and $i \in \{-1, 0, 1\}$. The maps can be described informally as follows: each $s \in S'$ acts on Y^+ in the way which most naturally extends the left-multiplication action $U \curvearrowright Y$; a flips the “v” attached to every vertex; while z flips only half the “v” shapes, because it flips the “v” attached to a vertex v_u if and only if d has an odd parity in u .

It is evident that the maps described above preserve adjacency in the one-skeleton of Y^+ , and hence determine isometries of Y^+ . Simple computations confirm that these definitions respect the relations in the presentation P_3 of G (some of the relations listed are vacuous). Therefore these definitions do indeed determine an isometric action $G \curvearrowright Y^+$. That the action is geometric follows easily from the fact that the left-multiplication action $U \curvearrowright Y$ is geometric. \square

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